J. Fluid Mech. (1996), *vol.* 322, *pp.* 355–382

Diffusion process produced by random internal waves

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(Received 12 October 1994 and in revised form 15 April 1996)

The aim of the paper is to present a new transport process which is likely to have great importance for understanding the internal constitution of the stars.

In order to set the problem in context, we first give a short presentation of the physical properties of the Sun and stars, described usually under the names *Standard Solar Model* or *Standard Stellar Models* (SSM). Next we show that an important shortcoming of SSM is that they do not explain the age dependence of the lithium deficiency of stars of known age: stars of galactic clusters and the Sun. It was suggested a long time ago that the presence of a macroscopic diffusion process in the radiative zone should be assumed, below the surface convective zone of solar-like stars. It is then possible for the lithium present in the convective zone to be carried to the thermonuclear burning level below the convective zone. The first assumption was that differential rotation generates turbulence and therefore that a turbulent diffusion process takes place. However, this model predicts a lithium abundance which is strongly rotation dependent, contrary to the observations. Furthermore, as the diffusion coefficient is large all over the radiative zone, it prevents the possibility of gravitational separation by diffusion and consequently leads to the impossibility of explaining the difference in helium abundance between the surface and the centre of the Sun. The consequence is obviously that we need to take into account another physical process.

Stars having a mass $M < 1.3M_{\odot}$ have a convective zone which begins close to the stellar surface and extends down to a depth which is an appreciable fraction of the stellar radius. In the convective zone, strong stochastic motions carry, at least partially, heat transfer. These motions do not vanish at the lower boundary and generate internal waves into the radiative zone. These random internal waves are at the origin of a diffusion process which can be considered as responsible for the diffusive transport of lithium down to the lithium burning level. This is certainly not the only physical process responsible for lithium deficiency in main sequence stars, but its properties open the way to a completely consistent analysis of lithium deficiency.

The model of generation of gravity waves is based on a model of heat transport in the convective zone by diving plumes. The horizontal component of the turbulent motion at the boundary of the convective zone is assumed to generate the horizontal motion of internal waves. The result is a large horizontal component of the diffusion coefficient, which produces in a short time an horizontally uniform chemical composition. It is known that gravity waves, in the absence of any dissipative process, cannot generate vertical mixing. Therefore, the vertical component of the diffusion coefficient is entirely dependent on radiative damping. It decreases quickly in the radiative zone, but is large enough to be responsible for lithium burning.

Owing to the radial dependence of velocity amplitude, the diffusion coefficient

increases when approaching the stellar centre. However, very close to the centre, nonlinear dissipative and radiative damping of internal waves become large and the diffusion coefficient vanishes at the very centre.

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1. Introduction

1.1. *Physical processes*

Interpreting observational data in astrophysics presents the difficulty of taking into account all the proper physical processes. This is a well-known problem in astrophysics, and there are many historical cases of enlightenment coming from taking into account in the theory of stellar bodies a forgotten physical process.

We shall analyse here a process of fluid dynamics which takes place inside the stars and has been almost entirely ignored by astrophysicists. We consider the properties of low-mass stars, having a mass close to the solar mass $M_\odot$: $M < 1.3\,M_\odot$. We first have a look at stellar structure; next we introduce the astrophysical problems raised by observational data and finally we show the need to take into account the diffusion process induced by random gravity waves.

1.2. *Stellar structure*

We first give a standard description of stellar structure, with the introduction of the appropriate technical terms. In *Standard Stellar Models* (SSMs) it is usually assumed that all motions are of small amplitude and in particular that the velocities are small compared to the sound velocity. We consider here non-rotating stars. A static non-rotating star has spherical symmetry. Going from the outside to the inside, we meet (figure 1):

(i) the surface layers, which constitute the *stellar atmosphere*. Down to a certain depth, these layers are stable. Despite the motions induced by fluid motions in the *convective zone* (next paragraph) we can consider as a first approximation that the stellar atmosphere is at rest.

(ii) Below the surface layers, stars with a mass $M < 1.3M_\odot$ have an unstable region (§2.1). In this region, the heat flux coming from the inside cannot be carried by radiation only and a large fraction of it is carried by convective motion. This unstable region, the *convective zone*, extends inside the star down to the radius where the heat flux can again be carried entirely by the radiation field. The depth of the convective zone depends on chemical composition and on stellar mass. The depth is relatively greater when the mass is smaller.

(iii) Below the convective zone, there is a stable *radiative zone*. Near the stellar centre, is the region of energy sources. Stars which have only experienced a short evolutionary phase have their energy produced by thermonuclear reactions taking place close to the centre, in a region called the *stellar core*. The rate of thermonuclear reactions can be expressed by simplified power-law expressions, T^n. In the solar case, with a central temperature around 15×10^6K, $n \simeq 4$, whereas, for stars of mass $M > 1.2\,M_\odot$, with a central temperature around 18×10^6K, the rate of energy production is more sensitive to temperature and we have $n \simeq 20$. In this case, heat production is more concentrated near the centre. The consequence is the presence of a convective core. We shall not consider this case in the present paper.

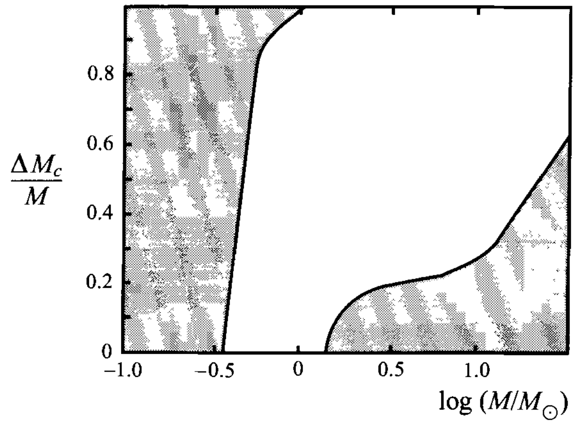

FIGURE 1. Size of the convective zone as a function of mass (Schatzman & Praderie 1990).

1.3. *Abundances of chemical elements*

The determination of the abundances of chemical elements in stellar atmospheres relies on both precise spectroscopic measurements and on elaborate stellar atmospheres models. The quality of the data has improved considerably during the last few years and very elaborate atmosphere models are available, for example those of Kurucz (1991). We can consider that at present data on the abundance of the elements are highly reliable.

1.4. *Chemical composition*

Stellar abundances are usually compared to *cosmic abundances*. These *cosmic abundances* are derived from chemical measurements of materials from Earth and of meteorites. We give here the usual notations. A logarithmic scale is used, the abundance of the element X being written

$$[X] = \log_{10}(X/\mathrm{Si}) + \log_{10}(\mathrm{Si}/H) + 12$$

where usually the abundance of silicium is assumed to be given by the relation $[\mathrm{Si}] - [\mathrm{H}] = -6$. With the reference value $[\mathrm{H}] = 12$ the cosmic abundance of lithium [Li] is usually assumed to be close to 3.3.

1.5. *Lithium*

We shall consider in this paper mainly the problem of lithium abundance.

Thirty years ago, solar abundances of chemical elements were considered essentially comparable to cosmic abundances. Perhaps the most important step has been the discovery by Herbig (1965) of a lithium abundance deficiency in solar-like stars. These stars with a mass $M < 1.3M_\odot$ are characterized by the presence of a surface convective zone.

Lithium is destroyed by the nuclear reaction

$$^7\mathrm{Li} + {}^1\mathrm{H} = 2\,^4\mathrm{He}$$

which begins to be important at a temperature of about 2.5×10^6K, much lower than the temperature in the central core. The absence of Li in the spectrum is an indication of lithium destruction. The presence of lithium lines in a spectrum means that the temperature at the bottom of the convective zone is lower than its burning temperature of 2.5×10^6K. It is an indication of stellar structure and properties of the outer layers, when the temperature T_b at the boundary of the convective zone is less than 2.5×10^6K.

Since 1965, the volume of data has considerably increased as shown by the most recent papers: Michaud & Charbonneau (1991), Soderblom *et al.* (1990, 1993*a*, 1993*b*), Soderblom (1991), Pasquini *et al.* (1994*a*), Pasquini, Liu & Pallavicini (1994*b*), Martin & Rebolo (1993), Randich & Pallavicini (1991), Balachandran (1993, 1994) but it appears also that they have become more and more difficult to understand. The simple idea that there is a universal time dependence of lithium abundance results in a number of inconsistancies which have led to the idea that it is necessary to take into account a variety of physical processes. Lithium burning, somewhere below the convective zone, appears to be the most important process. There is a clear age dependence of lithium abundance, the most obvious example being the remarkable case of the Hyades, but the physical conditions under which this is taking place implies a variety of processes or constraints.

1.6. *Physical processes*

Conditions of star formation. Stars are usually assumed to be formed by contraction of a cloud of interstellar matter. Before reaching the *main sequence*, they experience a number of physical processes, burning of ^{2}D in the central regions, mixing due to instabilities, mass loss due to stellar wind, etc. (Bodenheimer 1965; D'Antona & Mazzitelli 1984, 1994),

Stellar structure. The SSMs do not take into account all physical processes, such as *overshooting* from the unstable convective zone to the stable radiative zone, microscopic and macroscopic transport processes, the influence of plasma instabilities on the rate of thermonuclear reactions, etc. Schatzman (1969), Schatzman & Maeder (1981) and Schatzman *et al.* (1991) have shown the possible role of transport processes in SSMs, and Baglin & Lebreton (1990) their role in stellar evolution.

Metal abundance and activity. The depth of the convective zone depends on metal abundance, and consequently also on the efficiency of the dynamo mechanism which is at the origin of stellar activity. There are observational data (Pasquini *et al.* 1994*b*) which indicate the presence of these mechanisms, and Schatzman (1993*b*) has given preliminary indications of their effect on the amplitude of gravity waves.

Gravitational separation. This is an old problem (Aller & Chapman 1960; Michaud 1970; Vauclair & Vauclair 1982; Michaud & Profitt 1993), but new developments have come from the interpretation of the difference in abundance of ^{4}He between the solar convective zone and the radiative zone. This can be explained by gravitational separation of ^{1}H and ^{4}He (Christansen-Dalsgaard, Proffitte & Thomson 1993; Perez Hernàndez & Christensen-Dalsgaard 1994) but it requires a quiet solar interior, which contradicts the standard description of differential rotation and meridional circulation (e.g. Zahn 1992).

Analysis of the data. The data show no clear correlation between rotation and lithium deficiency, except that spindown and lithium deficiency are respectively associated with the time dependence of the velocity of rotation and of lithium burning (Balachandran 1993). Differential rotation and meridional circulation can generate a turbulent diffusion process, but it seems difficult to obtain a complete agreement between the data and the models (Baglin, Morel & Schatzman 1985; Schatzman & Baglin 1991; Montalban 1996).

1.7. *Models of lithium burning.*

For stellar masses smaller than about $1.3\,M_{\odot}$, the time dependence of lithium deficiency shows clearly that it is due to a relatively slow process, carrying lithium from the surface layers to the burning level which is located at the level where the temperature is about 2.5×10^6K. But at the same time, a large scatter in abundances for $(B - V) > 0.8$ (Soderblom *et al.* 1993*a*) shows the need to include several other physical processes, which have not yet been untangled.

In any case, dealing with transport process by diffusion from the bottom of the convective zone to the burning level is unavoidable (Schatzman *et al.* 1981), but the difficulty is in finding how this mechanism is generated. There are several possibilities and the major constraint is to build a model of the diffusive process which is physically fully consistent. Next come astrophysical constraints, such as the following: (i) the magnitude of the diffusion coefficient must satisfy the constraint of the lithium burning time scale, (ii) the model of lithium deficiency must be consistent with all data, and (iii) the diffusion mechanism must be consistent with all other physical properties of stars.†

1.8. *Turbulence.*

Turbulence seemed first to be the best candidate for generating the diffusion process (Schatzman *et al.* 1981) and Zahn (1983) provided the first model for the turbulent diffusion coefficient, turbulence being generated by meridional circulation and differential rotation. Baglin *et al.* (1985) and Schatzman & Baglin (1991) have shown that the model of turbulent transport by Zahn (1983) was not consistent with the mass dependence and the observed velocity of rotation. Zahn (1992) suggested a new model for the diffusion coefficient, where the main agent is the effect of the loss of angular momentum on differential rotation. The attempt by Charbonnel & Vauclair (1993) to explain the lithium abundances in the Hyades using the diffusion coefficient of Zahn (1992) rested on the assumption that all stars had the same initial velocity of rotation, $100\,\mathrm{km\,s^{-1}}$. This assumption ignores the existence of an initial velocity distribution function and therefore can be considered as a proof that the diffusion process depending on the total loss of angular momentum does not lead to an acceptable explanation of lithium deficiency.

In any case, all these turbulent diffusion coefficients in the radiative zone are derived from a phenomenological description which implies at least one free parameter, which needs to be adjusted for the Sun. But essentially the difficulty is consistency: as shown by Montalban (1996), the observed values of lithium abundance imply a distribution function of the velocity of rotation very different from the observed one (Bouvier 1994).

These difficulties led Schatzman (1991*a*, *b*, *c*) to take into account the diffusion process induced by gravity waves, which was considered for the first time in astrophysics by Press (1981) and Press & Ribicky (1981). Carruthers & Hunt (1986) have described the production of gravity waves by random motion in geophysical and astrophysical contexts. This model consider values of the ratio of the wave frequency to the Brunt–Väissälä frequency, ω/N, which do not correspond to the solar ratio.

† The existence of error bars is not ignored. But it should be noticed that some physical quantities are more critical than others. For example, the period of rotation of some stars in the Hyades are known with a great precision. But the error bar of lithium abundance can be larger. But if we consider the large dispersion of lithium abundances, for example in the Pleiades (Soderblom *et al.* 1993*a*), or in solar-like field stars (Pasquini *et al.* 1994*a*), it is clear that the observational data do not put a clear constraint on the diffusion process.

Schatzman (1991*a*, *b*, *c*, 1993*a*), following the presentation of Press (1981), gave only an order of magnitude estimate of the diffusion coefficient, but this gave a value of the lithium deficiency as a function of mass for the Hyades in agreement with the observations (Schatzman & Montalbàn 1995; Montalbàn & Schatzman 1996). As there was still a free phenomenological parameter, a more consistent demonstration of the expression for the diffusion coefficient was required.

1.9. *Structure of the paper*

As the model is based on the physical process due to gravity waves, we have to consider a model of the generation of gravity waves. In §2 we present a description of the motions in the convective zone using plumes, which allows boundary conditions between the radiative and the convective zone to be written. In §3 we give the properties of gravity waves and in §4 we derive the diffusion coefficient resulting from the presence of random gravity waves. This is applied in §5 to the lithium problem. In the conclusion, we consider the fact that the gravity waves model does not explain entirely lithium abundance properties and this suggests that it is necessary to take into account other physical processes.

2. Motions in the convective zone

2.1. *Introduction*

It is necessary to consider the nature of the motion in the convective zone in order to write the boundary conditions which determine the amplitude of internal waves in the radiative zone.

The standard description of the boundaries of the convective zone is the following. If we consider the adiabatic motion of a bubble, in a stable region a rising bubble becomes colder than the surroundings and comes back to its departure point. In SSMs, the stability condition is

$$\left(\frac{\mathrm{d}\log T}{\mathrm{d}\log P}\right)_{ad} \geqslant \left(\frac{\mathrm{d}\log T}{\mathrm{d}\log P}\right)_{rad}. \tag{2.1}$$

With the introduction of the Brunt–Väissälä frequency N:

$$N^2 = \frac{g}{H_P}\left(\frac{\mathrm{d}\log T}{\mathrm{d}\log P} - \left(\frac{\mathrm{d}\log T}{\mathrm{d}\log P}\right)_{ad}\right) \tag{2.2}$$

the stability condition is written

$$N^2 \geqslant 0. \tag{2.3}$$

However, when considering the dynamics of the motions in the convective zone, it is clear that there is inertial motion which carries the fluid beyond the standard boundary defined by equations (2.2) or (2.3). The presence of this penetration of the convective motion, or overshooting is important. The description of overshooting by Zahn (1991) provides an extension of the convective zone, but its size is not known. Note here that the model of Zahn depends of two phenomenological parameters: (i) the mean-square velocity at the boundary, and (ii) the asymmetry of the flow. With a proper choice of these parameters, there is a slight change in the solar model which leads (Berthomieu *et al.* 1993) to a better agreement with helioseismology data.

As we shall see in §3, the equation of propagation of a monochromatic internal wave without damping is a fourth-order differential equation in which the coefficient of the first derivative of the amplitude of the wave vanishes with $[(N^2/\omega^2)-1]$, where

ω is the gyrofrequency of the wave. This corresponds to a singularity of the solution in the WKBJ approximation. In a model like Zahn's, the Brunt–Väissälä frequency N presents a discontinuity at the boundary of the convective zone. Consequently, coming from the radiative region, the factor $[(N^2/\omega^2) - 1]$ does not vanish at the boundary, and this makes writing boundary conditions easier (Schatzman 1993; Montalbàn 1994). It appears that it is equivalent to the conditions described by Press (1981) in the case of a steep variation of N^2, and it can be simply assumed that there is continuity of the horizontal component of the turbulent velocity at the boundary of the convective zone.

At this point, everything depends on the description of the velocity field at the boundary of the convective zone. The mixing length theory (MLT) of the convective zone is usually used to describe stellar models. Despite the fact that the MLT leads to stellar models essentially in agreement with observational data, it should be kept in mind that, from the hydrodynamical point of view, the MLT has a number of inconsistencies. We present briefly the possible models.

2.1.1. *Kolmogorov spectrum*

The simplest idea consists in assuming the presence in the convective zone of a Kolmogorov spectrum (known as K41), with a characteristic horizontal scale l_H, and a maximum velocity provided by the MLT of the convective zone (Schatzman 1991*a*,*b*,*c*, 1993).

2.1.2. *Bumps*

It is possible to use the model used by meteorologists (Townsend 1965; Stull 1976) for the production of gravity waves in the stable region above the atmospheric convective zone. It assumes that gravity waves are produced by velocity bumps of a Gaussian shape. What are the properties of gravity waves generated this way? Do they maintain a sufficiently large amplitude to produce the expected diffusion effects? Taking into account the damping of propagating internal waves by dissipative effects, Townsend (1965) gives the following characteristic damping depth z_C:

$$z_C = \frac{v_0^3 \omega}{D N^3}, \tag{2.4}$$

where v_0 is the velocity at the boundary, ω the angular frequency, N the Brunt–Väissälä frequency, and D the diffusion coefficient. Townsend gives a penetration of gravity waves into the Earth's stable atmosphere over the cloud tops of the order of 100 metres. But in the stellar case, in order to take into account the spectral properties of the generating mechanism of internal waves, it is better to use the similar expression for z_C given by Press (1981):

$$z_C = \frac{\omega^4}{D k_H^3 N^3}, \tag{2.5}$$

where k_H is the horizontal wavenumber. The horizontal wavelength is assumed to be of the order of the vertical scale height. For the lowest frequency ω_0,

$$\omega_0 = k_H v_0, \tag{2.6}$$

we obtain a very small vertical scale z_C of the order of 10^7 cm, and such a value would rule out immediately the idea of any effect of internal waves. However, we can consider the fact that turbulence has a large spectrum of frequency and wavenumber since ω, for a Kolmogorov spectrum, is proportional to $k^{2/3}$. For $k \approx 10 k_H$ the ω^4

term is about 500 times bigger and damping has a characteristic vertical scale of the order of 10^{10} cm. The corresponding characteristic velocity is smaller, by only a factor $10^{-1/3}$, but penetrates deeper. It appears clearly that if we take into account the statistical properties of the turbulent spectrum (Schatzman 1993; Montalbàn 1994) parts of the gravity waves will penetrate deeply inside the Sun.

2.1.3. *Plumes*

Numerical simulations (Hurlburt, Toomre & Massaguer 1986; Steffen 1993; Steffen & Freytag 1991; Cattaneo & Malagoli 1992) suggest that the plume model of convective heat transport is probably closer to physical reality than bumps. We follow here the simple model of Rieutord & Zahn (1995), which is based on the experimental data obtained in the Earth's atmosphere. The properties of the turbulent velocity in the plumes are also given by experimental data (List 1982), and the only free parameter is the total number of plumes at the surface of the Sun, which is estimated by Rieutord & Zahn (1995) to be of the order of 1000. They occupy 21% of the surface of the Sun at the bottom of the convective zone.

2.2. *The plumes model*

The plumes model of Rieutord & Zahn (1995) is certainly a simplified picture. However, it appears to be a good introduction to a description of the production of internal waves. To obtain better numerical values, we extend the model of Rieutord & Zahn to the spherical case, and give a model of the turbulent motion generated in a plume on its arrival at the boundary of the convective zone (Schatzman & Montalbàn 1995).

2.2.1. *Plane parallel case*

We write first the equations describing a plume in the plane parallel case, following the model of Rieutord & Zahn, before applying the model to the spherical case. A plume broadens during its propagation. In the most simple model, the gas surrounding a plume is almost at rest. Consequently, entrainment of the gas into the plume is due to the horizontal velocity gradient. It is assumed that the plumes are axisymmetric and that their horizontal radius $b = b(r)$ is small compared to the radius r:

$$b(r) \ll r. \tag{2.7}$$

It is then possible to introduce, instead of the spherical coordinate θ, colateral with the symmetry axis of the plume, the distance $s = r\theta$ to the symmetry axis. In the same way, instead of $\partial/r\partial\theta$, we can write $\partial/\partial s$. It is then possible to follow the model of Rieutord & Zahn (1995) and to use the formal representation by Gaussians of the vertical component of the velocity v_r, the density fluctuations $\delta\rho$, and of the enthalpy excess δh:

$$v_r(r,t) = V(r)\exp\left(-s^2/b^2\right), \tag{2.8}$$

$$\delta\rho(r,s) = \Delta\rho(r)\exp\left(-s^2/b^2\right), \tag{2.9}$$

$$\delta h(r,s) = \Delta h(r)\exp\left(-s^2/b^2\right). \tag{2.10}$$

We can then write the three equations governing the motion:
mass conservation

$$\frac{1}{r^2}\frac{\partial}{\partial r}r^2\rho v_r + \frac{1}{s}\frac{\partial}{\partial s}s\rho v_s = 0; \tag{2.11}$$

conservation of momentum

$$\frac{1}{r^2}\frac{\partial}{\partial r}r^2\rho v_r^2 + \frac{1}{s}\frac{\partial}{\partial s}s\rho v_r v_s = \delta\rho g; \tag{2.12}$$

conservation of energy

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(\delta h + \tfrac{1}{2}v^2\right) r^2\rho v_r + \frac{1}{s}\frac{\partial}{\partial s}\left(\delta h + \tfrac{1}{2}v^2\right) s\rho v_s = 0, \tag{2.13}$$

which assumes that the radiative flux inside the plumes is negligible. Rieutord & Zahn (1994) integrated these three equations with respect to s from zero to infinity. The value of each integral is determined by the entrainment hypothesis of Taylor (Turner 1986; Morton, Taylor & Turner 1956). It is assumed that the radial flux of matter towards the axis of the plume is proportional to the vertical velocity on the plume axis:

$$\lim_{s\to+\infty} s v_s = -\alpha b(r)\left|v_r(r,0)\right|, \tag{2.14}$$

where α is a measure of the entrainment magnitude. With the notation

$$\xi = 1 + \Delta\rho/\rho \tag{2.15}$$

we obtain the following equations:
mass conservation

$$\frac{1}{r^2}\frac{\partial}{\partial r}r^2 V(r) b^2 \rho_0 \frac{\xi+1}{2} = 2\alpha b(r)\rho_0 V(r); \tag{2.16}$$

conservation of momentum

$$\frac{1}{r^2}\frac{\partial}{\partial r}r^2 b^2 V^2(r)\rho_0\frac{2\xi+1}{2} = 2gb^2 2\Delta\rho; \tag{2.17}$$

conservation of energy

$$\Delta h\rho_0 V(r) b^2\frac{2\xi+1}{6} + V^3(r)\rho_0 b^2\frac{3\xi+1}{12} = -\frac{\mathscr{F}}{\pi}\frac{r_b^2}{r^2}; \tag{2.18}$$

where r_b is the radius of the lower boundary of the convective zone and $\mathscr{F}$ is the total flux (enthalpy flux plus kinetic energy flux) carried by the plume.

The size of the plume given by its radius b at the bottom of the convective zone of radius r_b is

$$b = \beta_0(R - r_b) \tag{2.19}$$

with $\beta_0 = (6\alpha/7)$, and $\alpha = 0.083$ (Turner 1986). This gives $b = 14500$ km. With the total number of plumes on the Sun $N_{Pl} = 1000$, this corresponds, as mentioned above, to occupation by plumes of 21% of the solar surface. The maximum vertical velocity of a plume at the bottom of the convective zone is

$$V = \left(\frac{8(L/N_{Pl})}{\pi\rho_b\beta_0^2 z_0^2}\right)^{1/2}. \tag{2.20}$$

With $\rho_b = 0.2\,\mathrm{g\,cm^{-3}}$, the solar luminosity and $z_0 = 200\,000$ km,

$$V = 2.84\;10^4\,\mathrm{cm\,s^{-1}} \tag{2.21}$$

which is much larger than the value derived from the MLT,

$$V_{MLT} = \left(\frac{\varphi L}{4\pi r^2\rho_{00}}\right)^{1/3} = 3.93\;10^3\,\mathrm{cm\,s^{-1}} \tag{2.22}$$

with $\varphi = 0.1$.

2.2.2. *Spherical case*

To obtain the best values of the diffusion coefficient, we have to take into account the effects of sphericity, which are not included in the plume model of Rieutord & Zahn (1995) and furthermore to use a reasonable description of the turbulence. We shall assume that the diameter of the plumes is small compared to the stellar radius, which allows use of equations (2.8), (2.9), (2.10).

(i) For diving plumes, in order to take into account the effect of sphericity we modify the self-similar solution of Rieutord & Zahn by introducing correction terms for the width of the plumes, their velociy and their density. After integration, this gives for the width

$$b_1 = 0.92b \tag{2.23}$$

and for the velocity

$$V_1 = 1.34V. \tag{2.24}$$

(ii) Turbulence is assumed to be generated by shear flow at the boundary of a plume. We define the boundary as where the shear is maximum, $s_1 = 1/\sqrt{2}b$. We assume that the maximum scale of the turbulence b_M is defined as where the velocity gradient is maximum. This gives

$$b_M = \left(\tfrac{1}{2}\mathrm{e}\right)^{1/2} b_1. \tag{2.25}$$

To define the characteristic velocity of the turbulent flow we use the experimental results for jets from List (1982):

$$\langle u^2\rangle^{1/2} = 0.285\langle V_1\rangle \equiv V_M. \tag{2.26}$$

We define the average velocity V_{av} inside the radius of maximum shear as

$$\langle V_{av}\rangle = \langle V\rangle \int_0^{s_1} \mathrm{e}^{-s^2/b^2} s\mathrm{d}s \Big/ \int_0^{s_1} s\mathrm{d}s. \tag{2.27}$$

The aim of this description of the turbulent flow in plumes is to avoid, if possible, the introduction of adjustable phenomenological parameters.

2.3. *Penetrative convection*

Owing to the change of sign of N^2 beyond the Schwarzschild limit, the buoyancy force changes sign and generates an upwards motion. Figure 2 gives a qualitative view of a plume profile. We know from numerical models that the vertical motion back into the convective zone is accompanied by a horizontal extension of the plume. We are interested in the properties of this horizontal extension as it is the location of turbulent motions. The theory of horizontal extension for motion in a fluid (Aseda & Imberger 1993; Larson & Jönsen 1995) is at present being applied by Lo (1996) to the astrophysical case

The plume model is based on a self-similar Gaussian representation of velocities (Rieutord & Zahn 1995). In this model, turbulent velocity in the plume decreases horizontally like $\exp\{-(s/b)^2\}$, where s is the distance to the vertical axis of revolution of the plume, and the auto-correlation function decreases very fast, away from the axis, beyond the distance $b(r)$, which is the basis of the description of the plumes. With the Gaussian auto-correlation function, the contribution of the small wavenumbers k to the diffusion process turns out to be negligible. In order to obtain the average auto-correlation function, it is necessary to take into account the properties of the velocity field in the penetration region. Even if we assume that the auto-correlation function has locally a Gaussian shape, we need to describe the average properties.

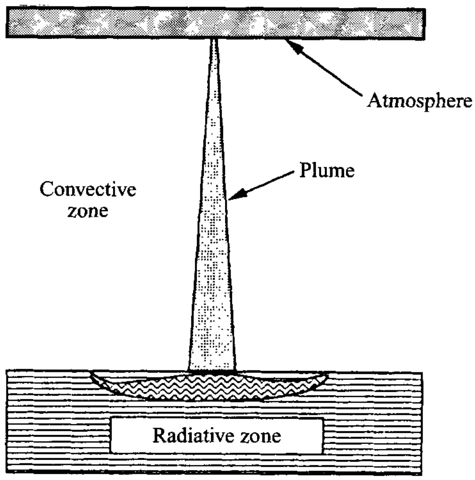

FIGURE 2. The plume flow. After penetration there is an area of upwards fluid motion which is sketched here.

This can be done by taking the Fourier transform of local auto-correlations functions and then taking their average over all horizontal distributions of the properties of the velocities.

The velocity in the overshooting region decreases with the distance to the symmetry axis of the plume and we choose to describe the properties of the velocity by a power law, $v_s \sim s^{-n}$. As we shall explain §4, the result is an average auto-correlation function of the turbulent velocity which is given, in a first approximation, by a power law $(s' - s)^n$. The s^{-1} law for the amplitude of the velocity in the spreading flow, just mentioned, suggests taking, as an approximation of its Fourier transform, $(k/k_M)^{-2}$. A faster decrease of the auto-correlation at large distances suggests taking, as an approximation of its Fourier transform, $(k/k_M)^{-3}$.

We do not yet have a complete theory of the auto-correlation at large distances from the plume axis, and we need to introduce phenomenological parameters. We shall show §4 how to use this description of the turbulent flow to write the boundary conditions leading to the formation of random gravity waves.

3. Internal waves

3.1. *Introduction*

Gravity modes can be present as standing waves or propagating waves. As standing waves, they have been considered for a long time as eigen-modes of stellar oscillations. Since the start of helioseismology, there has been a search for low-frequency gravity modes as solar pulsations in the Sun but so far it has been unsuccessful. Gough (1991) and de Rujula & Glashow (1992) have considered the contribution of these modes to solar structure and their effect on the production of solar neutrinos. Bahcall & Kumar (1993) have shown that the effect is negligible.

We do not consider eigen-modes here but instead, as suggested by Press (1981) and Press & Ribicky (1981), the whole spectrum of propagating gravity waves generated by the turbulent flow at the boundary of the convective zone. As we shall see, these waves are damped, due to radiative thermometric diffusivity. Going downwards from the boundary of the convective zone, the amplitude of internal waves first decreases.

On moving towards the stellar centre, the amplitude increases but finally radiative damping becomes so large that the amplitude goes to zero. This can be compared to the results of Carruthers & Hunt (1986), where the radiative damping does not have the same importance.

3.1.1. *Nonlinear effects*

Nonlinear effects are almost negligible if the amplitude of the horizontal motion is small compared to the horizontal scale of the motion. As an order of magnitude, this condition can be written

$$k_H > (D_H \tau)^{-1/2}$$

where D_H is the horizontal diffusion coefficient, τ the characteristic auto-correlation time and k_H the horizontal wavenumber, and this gives $k_H > 7 \times 10^{-11}$ which is obviously fulfilled close to the boundary of the radiative zone, as it corresponds to dimensions smaller than the solar radius. As the horizontal diffusion coefficient (4.6) decreases like ρ^{-1}, close to the boundary the linear approximation condition is fullfilled. However, when approaching the centre, the r^{-3} term becomes important, and it is necessary to take into account damping processes (§5.2). It is found that nonlinear effects clearly never become important.

Wave dissipation is such that it is not necessary to take into account the reflection of the waves near the centre and boundary conditions can be ignored. The Brunt–Väissälä frequency becomes equal to the wave frequency only very close to the centre ($r < 0.01 R_\odot$) and the change from propagating waves to evanescent waves can be ignored.

3.1.2. *Effects of internal waves*

Two effects due to internal waves have to be considered:

Macroscopic diffusion process. Through a process similar to what takes place in a turbulent flow (Batchelor 1952; Knobloch 1977), random internal waves generate a diffusion process. However, there is a difference between a perfect fluid (no dissipative process) and a fluid presenting a dissipative mechanism.

Consider first a perfect fluid. As explained by Press (1981), when the thermometric diffusivity is zero, the diffusive transport is entirely horizontal (Bretherton 1969; Grimshaw 1984; McIntyre 1973). There is no vertical diffusive transport as the fluid entropy carries a 'memory' of the level at which it should sink or rise, counteracting any second order fluid forces.

Consider now non-conservation of entropy resulting from radiative heat transfer. Irreversible effects, due to dissipative processes, generate a finite r.m.s displacement of fluid elements. A diffusive process takes place, and the aim of the present paper is to obtain the best possible description, with important consequences for the surface abundance of lithium and beryllium (Montalbàn 1994) and possibly for the solar neutrino flux.

Transport of angular momentum. Internal waves carry angular momentum (Goldreich & Nicholson 1989*a,b*) and a description of stellar rotation has to take this effect into account. A preliminary approach to this process has been given by Schatzman (1993), and it will not be considered in the present paper.

3.2. *Amplitude as a function of depth*

A complete description of gravity waves has been available for a long time (Cowling 1941; Ledoux & Walraven 1958). We are interested here in the classical case, when the presence of pressure waves can be ignored, and we assume that the fluid motion can be described in the linear approximation by the fourth-order differential equation given by Press (1981). Introducing the function ψ related to the vertical component of the fluid velocity u_V by the relation

$$u_V = \psi \rho^{1/2} k_H^2, \tag{3.1}$$

we have

$$\frac{\partial^2 \psi}{\partial r^2} + \left(\frac{N^2}{\omega^2} - 1\right) k_H^2 \psi + \frac{\mathrm{i} D_{th}}{\omega} \left(\frac{\partial^2}{\partial r^2} - k_H^2\right)^2 \psi = 0. \tag{3.2}$$

Assuming that the damping effect is small, it is possible to replace the imaginary part of the equation for the function ψ by the solution of the equation without damping. Then, the WKB approximation provides the expression for the fluid velocity of propagating internal waves as a function of depth (Press 1981). For a wave propagating downwards, the vertical component of velocity is

$$\begin{aligned} u = u_{Hb} & \left(\frac{N_b^2}{\omega^2} - 1\right)^{-1/2} \left(\frac{r_b}{r}\right)^{3/2} \left(\frac{\rho_b}{\rho}\right)^{1/2} \left(\frac{N^2}{\omega^2} - 1\right)^{-1/4} \\ & \times \left(\frac{N_b^2}{\omega^2} - 1\right)^{1/4} \exp\left(\mathrm{i} \int k_V \mathrm{d}r - \mathrm{i}\omega t\right) \\ & \times \exp\left[\frac{1}{2} \int \left(\frac{D_{th} k_H^2}{N}\right) \left(\frac{N}{\omega}\right)^4 \left(\frac{N^2}{\omega^2} - 1\right)^{-1/2} \frac{N}{\omega} k_H \mathrm{d}r\right]. \end{aligned} \tag{3.3}$$

The index b refers to the boundary of the convective zone. For example, u_{Hb} is the horizontal component of the velocity at the boundary. The relation between the vertical wavenumber k_V and the horizontal wavenumber k_H is

$$k_V^2 = k_H^2 \left(\frac{N^2}{\omega^2} - 1\right) \tag{3.4}$$

where N is the Brunt–Väissälä frequency. Note that as long as $N^2/\omega^2 > 1$ we have a progressive wave which turns into an evanescent wave when $N^2/\omega^2 < 1$. In most of the radiative zone, $N^2/\omega^2 \gg 1$. In the radiative zone, a characteristic property of gravity waves is damping due to radiative dissipation (figure 3). We write the coefficient expressing the effect of damping on the vertical velocity as $\exp\left[-\frac{1}{2}A\right]$. As the horizontal wavenumber k is proportional to $1/r$, it is necessary to introduce the boundary wavenumber k_b. Then, we have

$$A = f k_b^3 / \omega^4 \tag{3.5}$$

with

$$f = \int_r^{r_b} D_{th} N^3 \left(\frac{r_b}{r}\right)^3 \left(1 - \frac{\omega^2}{N^2}\right)^{-1/2} \mathrm{d}r. \tag{3.6}$$

Taking this to the second order, the approximation appears to be valid as long as

$$\frac{D_{th}}{\omega} k_H^4 \left(\frac{N^2}{\omega^2} - 1\right)^4 \ll \left(\frac{N^2}{\omega^2} - 1\right) k_H^2. \tag{3.7}$$

Denoting x as the fraction of the radius r_b of the boundary of the convective zone,

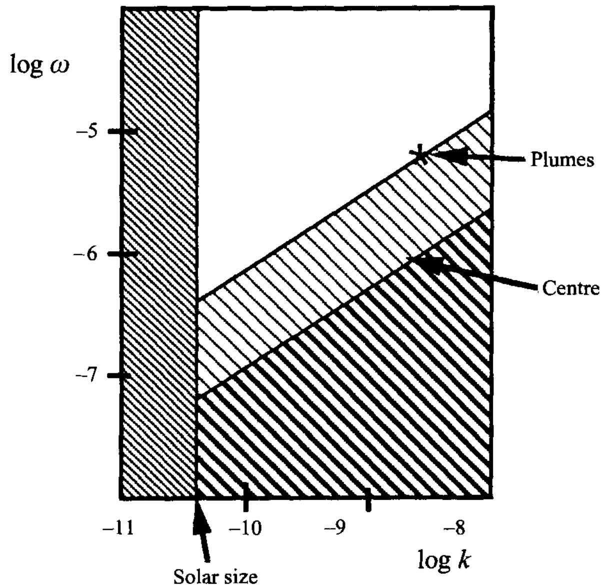

FIGURE 3. Domain of weak damping of gravity waves near the boundary of the convective zone and near the solar centre. The wavenumber is the abscissa and the gyrofrequency is the ordinate.

we have with $r \sim (1/k_H)$,

$$k_H = \frac{1}{x} k_{Hb}. \tag{3.8}$$

When $N^2/\omega^2 \gg 1$, the condition given by equation (3.7) can be written

$$x^2 \gg \frac{k_{Hb}^2 N^2 D_{th}}{\omega^3}. \tag{3.9}$$

With the characteristic wavenumber k_B and the characteristic frequency of the turbulence at the boundary, $\omega_B = k_B u$, we obtain the condition

$$x^2 \gg \frac{N^2 D_{th} b}{2\pi u^3}. \tag{3.10}$$

In order to obtain a quantitative estimate of condition (3.10), we can use characteristic quantities given by the plume model (see §2.2). We define the wavenumber which corresponds to the plume radius b at the boundary of the convective zone by

$$k_B = 2\pi/b. \tag{3.11}$$

With the values given in §2.2 of

$$u = 1.085.10^4 \text{ cm s}^{-1} \text{ and } b = 1.555 \times 10^9 \text{ cm} \tag{3.12}$$

and an order of magnitude for the other quantities:

$$N^2 \simeq 4 \times 10^{-6} \text{ and } D_{th} \simeq 10^7 \tag{3.13}$$

we have the condition

$$x \gg 0.088. \tag{3.14}$$

It is clear that since the damping of monochromatic waves depends on k_b and ω, the result will depend on the integration over the k_b and ω spectra. The main problem is then to describe the production of gravity waves.

3.3. *Complete description of the damping*

It is necessary to consider more completely the solution of equation (3.2). With the definition of x (3.8) and the notations

$$\frac{\omega r_b^2}{D_{th}} = \delta, \qquad \frac{4\pi^2 r_b^2}{b^2 x^2} = K^2, \tag{3.15}$$

and introducing a factor h to be taken later equal to 1, equation (3.2) becomes

$$\frac{\partial^4 \Psi}{\partial x^4} + \left[-\mathrm{i}\delta - 2K^2\right] h^2 \frac{\partial^2 \Psi}{\partial x^2} + \left[-\mathrm{i}\delta K^2 \left(\frac{N^2}{\omega^2} - 1\right) K^4\right] h^4 \Psi = 0 \tag{3.16}$$

and we proceed with the technique used for WKB solutions. We write

$$\Psi = \exp \int \varphi \mathrm{d}x \tag{3.17}$$

with

$$\varphi = h\varphi_0 + \varphi_1 + h^{-1}\varphi_2 + \ldots \tag{3.18}$$

and we obtain the equations

$$\left.\begin{aligned} &\varphi_0^4 + \left[-\mathrm{i}\delta - 2K^2\right] \varphi_0^2 + \left[-\mathrm{i}\delta K^2 \left(\frac{N^2}{\omega^2} - 1\right) + K^4\right] = 0, \\ &4\varphi_0^3\varphi_1 + 6\varphi_0^2\varphi_0' + \left(-\mathrm{i}\delta - 2K^2\right)\left(\varphi_0' + 2\varphi_0\varphi_1\right) = 0. \end{aligned}\right\} \tag{3.19}$$

The solution of equations (3.19) is

$$\left.\begin{aligned} \varphi_0^2 &= \tfrac{1}{2}\left[\mathrm{i}\delta + 2K^2 - \left(-\delta^2 + 4\mathrm{i}\delta K^2 N^2/\omega^2\right)^{1/2}\right], \\ \varphi_1 &= -\frac{\varphi_0'}{\varphi_0}\frac{6\varphi_0^2 - (\mathrm{i}\delta + 2K^2)}{4\varphi_0^2 - 2(\mathrm{i}\delta + 2K^2)}. \end{aligned}\right\} \tag{3.20}$$

Introducing the function $\sinh\xi$, characteristic of the radiative damping of the wave with a vertical wavenumber $(N/\omega)(k_b/x)$, defined by

$$\sinh\xi = \frac{4k_b^2 N^2 D_{th}}{\omega^3 x^2}, \tag{3.21}$$

we obtain the real and the imaginary parts of φ_0, called a and c respectively:

$$a = \left\{\frac{\delta}{2}\left(\frac{\omega^2}{4N^2}\sinh\xi - \frac{1}{2}\sinh\frac{\xi}{2} + \left[\left(\frac{\omega^2}{4N^2}\sinh\xi - \frac{1}{2}\sinh\frac{\xi}{2}\right)^2 + \sinh^4\frac{\xi}{4}\right]^{1/2}\right)\right\}^{1/2} \tag{3.22}$$

and

$$c = -\frac{1}{a}\frac{\delta}{2}\sinh^2\frac{\xi}{4}. \tag{3.23}$$

When ξ is small, i.e. when

$$\frac{4k_b^2 N^2 D_{th}}{\omega^3 x^2} \ll 1 \tag{3.24}$$

the expressions for a and c reproduce the solution of Press (1981). When we replace, in condition (3.24), k_b and ω respectively by the quantities which are characteristic of plumes, $2\pi/b = 4.49 \times 10^{-9}$ and $\omega_B = v/b = 5.71 \times 10^{-6}$ we find, close to the boundary of the convective zone

$$\sinh \xi = \frac{17.32}{x^2} \gg 1. \tag{3.25}$$

In fact, as we shall see §4, in order to calculate the diffusion coefficient, we have to carry out an integration over the whole spectrum of the perturbation, which means introducing the Fourier transform and integrating the wavenumber k_b from $-\infty$ to $+\infty$. This is valid in the plane parallel approximation. In fact, as we are considering a sphere, we have to deal with a summation over spherical harmonics. Replacing the summation by an integration over a continuous variable, the wavenumber, this would be equivalent to carry the integration from $2\pi/r_b$ to infinity. In that case, keeping the same characteristic frequency, equation (3.25) becomes

$$\sinh \xi = 0.0136/x^2 \tag{3.26}$$

which means that for $x \gg 0.1$ we can use the equations resulting from the approximation ξ small.

4. Diffusion coefficient

4.1. *Ensemble average*

The classical expression for the diffusion coefficient in a turbulent flow has been given by Taylor (1921). A more general demonstration has been given by Knobloch (1977, 1991). When the diffusion process is due to a random velocity field, the diffusion coefficient is obtained by taking an ensemble average.

4.2. *Horizontal diffusion*

As recalled by Press (1981), Bretherton (1969) has shown that adiabatic motion brings any fluid element back to the level from where it started. Vertical diffusion therefore vanishes in the adiabatic case, whereas horizontal diffusion is always present.

For horizontal diffusion, the expression given by Knobloch (1977, 1991) is valid:

$$D_H = \int_0^\infty \langle v_H(t) v_H(t') \rangle \mathrm{d}(t'-t) \tag{4.1}$$

where the angle brackets denote an ensemble average. We shall first consider the plane parallel case, and then give an approximation for the spherical case.

In any case, it is necessary to express the relation between the ensemble average of the second order product of horizontal velocities in the radiative zone $\langle u_H(x,t) u_H(x,t') \rangle$ described by internal waves and the ensemble average of the square of the turbulent velocity $\langle u_T(x,t) u_T(x,t') \rangle$ which generates internal waves. In order to obtain this relation, we take the inverse Fourier transform,

$$u_H(x,t) = \int\int\int \hat{u}_H(l,m,\omega) \exp\{-\mathrm{i}lx - \mathrm{i}ly - \mathrm{i}\omega t\}\, \mathrm{d}l\, \mathrm{d}m\, \mathrm{d}\omega \tag{4.2}$$

together with a similar one for the turbulent velocity. Correspondingly, $\hat{u}_H$ is the Fourier transform of the sum of the motions u_T due to N_{Pl} plumes (see §3.3)

To calculate the ensemble average we have to write the relation between two ensemble averages:

$$\langle \hat{u}_T(l,m,\omega) \hat{u}_T(l',m',\omega') \rangle \text{ and } \langle \hat{u}_H(l,m,\omega) \hat{u}_H(l',m',\omega') \rangle. \tag{4.3}$$

We have to take into account the combination of the correlated noise inside the plumes and the white noise coming from all the plumes together. As shown by Chandrasekhar (1943), Rayleigh (1880, 1899) and recalled by Garciá-Lopez & Spruit (1991), as an effect of white noise interference, the second ensemble average (internal waves) will arise from the sum of the auto-correlation function $\mathscr{A}$ at the bottom of each plume:

$$\sum_{i=1}^{i=N_{Pl}} \mathscr{A}((x-x_i)^2+(y-y_i)^2) \tag{4.4}$$

If the auto-correlation function is a Gaussian, $\exp\big(-(x^2+y^2)/b^2)\big)$, its Fourier transform is $\exp(-\frac{1}{4}b^2k^2)$. In the same way, the Fourier transform of the auto-correlation function in time is $\exp(-\frac{1}{4}\tau^2\omega^2)$.

We can then obtain the horizontal diffusion coefficient D_H,

$$D_H = \int \langle u_H(t)u_H(t')\rangle \mathrm{d}(t'-t).$$

This is the result of taking the Fourier transform in the plane parallel case with horizontal coordinates going from$-\infty$ to $+\infty$. In the spherical case we should use spherical harmonics, but as we only want orders of magnitude we use the following approximations: (i) we replace the infinite horizontal surface by a square with a side $2L$, (ii) we replace the sum of trigonometric functions by an integral from $-L$ to $+L$. Which value should we choose for L ? Owing to the finite size of the source there is a minimum wavenumber k_{min}, which is the wavenumber of the largest scale and is of the order of $2/r_b$.

The average squared velocity is equal to the squared velocity in the area of a plume times the ratio of the area occupied by the plumes, $N_{Pl}\pi r b_M^2$ divided by the area of the spherical bondary $4\pi r_b^2$. Because $b_M\, k_{min} \ll 1$, and using the classical approximate expression for the diffusion coefficient due to random motion

$$D = \int_0^\infty \langle u(t)u(t')\rangle \mathrm{d}(t'-t) \simeq u^2\tau,$$

where τ is the auto-corelation time scale, we have at the boundary

$$D_H \simeq N_{Pl}\,\frac{b_M^2}{r_b^2} v_T^2 \tau. \tag{4.5}$$

The order of magnitude of τ is $2\pi b/v_b$ where b is the horizontal correlation length, v_b is the horizontal turbulent velocity at the boundary, and N_{Pl} the number of plumes over the surface of the boundary of radius r_b. With the values of the scale b_M given by equation (2.22) and of the velocity given by equation (2.23) this provides an horizontal diffusion coefficient of the order of $5 \times 10^{11}\mathrm{cm}^2\,\mathrm{s}^{-1}$ at the boundary, which means a mixing time scale of the order of 500 years.

Taking into account the depth dependence of radiative damping, the average contribution of all plumes over the surface of the boundary and the interference effect, we can obtain the horizontal diffusion coefficient as a function of depth. With the method of steepest descent for integration over ω, we have

$$D_H \simeq N_{Pl}\frac{b_M^2}{4r_b^2}\tau u_T^2 \exp\left[-\frac{3}{2}\left(\frac{f k_{min}^3 \tau^4}{8}\right)^{1/3}\right]\left(\frac{\rho_b}{\rho}\right)\left(\frac{r}{r_b}\right)^{-3}. \tag{4.6}$$

The result is very sensitive to the values of k_{min} and τ. With the values obtained in §2 the diffusion coefficient slowly decreases as a function of depth. Around $r/r_b \simeq 0.1$ it gives a mixing time scale of the order of 5000 years, which is very short compared to evolutionary time scales. We can conclude that horizontal mixing is very efficient and generates a uniform horizontal chemical composition.

4.3. *Vertical diffusion*

In the presence of radiative damping, there is no entropy conservation, and there remains a vertical diffusion coefficient remains. We start from the Lagrangian expression given by Frisch (1987) derived from the classical treatment of stochastic differential equations,

$$D_V = \int_0^\infty \langle v(t)v(t')\rangle \mathrm{d}(t'-t) \tag{4.7}$$

where the angle brackets denote ensemble average. Using the quantities A (equation (3.5)), and

$$f = \int_r^{r_b} D_{th} N^3 \left(\frac{r_b}{r}\right)^3 \mathrm{d}r \tag{4.8}$$

we can express the Lagrangian velocity as

$$v_z = u(z,y) + \frac{\partial u}{\partial z}\int^t u \mathrm{d}t. \tag{4.9}$$

We now use expression (4.9) for the Lagrangian velocity in terms of the Eulerian (3.3) velocity where the index b designates the values at the boundary of the convective zone; k_b is the horizontal wavenumber:

$$k_b^2 = l_b^2 + m_b^2. \tag{4.10}$$

Since the horizontal wavenumber is proportional to $1/r$ and the horizontal coordinate is proportional to r, the products lx and my do not depend on r and the sum $(lx+my)$ can be written equivalently as $(l_b x_b + m_b y_b)$, where x_b and y_b are the horizontal coordinates at the boundary. We can now drop the index b.

For $N^2/\omega^2 \gg 1$, which is the case in almost the whole radiative zone and is also true at the upper boundary if penetrative convection (Zahn 1991) has been taken into account, the expression (3.3) for the vertical velocity simplifies to

$$u_z(x,y,r,l,m,\omega) = u_{Hb}(l,m,\omega)\frac{|\omega|}{N_b}\left(\frac{r_b}{r}\right)^{3/2}\left(\frac{\rho_b}{\rho}\right)^{1/2}\left(\frac{N}{N_b}\right)^{-1/2} \times \exp\left[\mathrm{i}\left(\int k_V \mathrm{d}r - \omega t\right) + \mathrm{i}(lx+my)\right]\exp\left[-\frac{A}{2}\right]. \tag{4.11}$$

According to Bretherton's (1969) theorem, the first-order term makes a zero contribution to the diffusion process. Therefore we keep in the z-derivative only the contribution due to radiative damping. We have

$$v_z = -\frac{k^3}{\omega^4}\frac{1}{2}\frac{\partial f}{\partial z}u_z\frac{u_z}{\mathrm{i}\omega} \tag{4.12}$$

The diffusion coefficient now takes the form

$$D_{Mix} = \int_0^\infty \left\langle \frac{\partial u_z(t)}{\partial z}\int^t u_z \mathrm{d}t \quad \frac{\partial u_z(t')}{\partial z}\int^{t'} u_z(t') \right\rangle \mathrm{d}(t-t') \tag{4.13}$$

which turns out to be of the fourth order with repect to the velocity.

As already mentioned, we are concerned with the problem of the generation of internal waves by turbulent motion inside plumes, at the boundary of the convective zone. We base the description of the plumes on the picture given by Rieutord & Zahn (1995). We use first the plane parallel approximation as a step towards the description of motions with spherical symmetry.

As the propagation of internal waves is described by monochromatic waves, equation (3.3), it is necessary to introduce the Fourier transform of the motion occurring at the boundary of the radiative zone and then to take the inverse Fourier transform at the depth z, including this time k-dependent terms introduced by the z derivatives present in equation (4.13). In order to take the ensemble average, we follow the rule given by Knobloch (1977), for the products of two ensemble averages of two terms. This results in the presence of three products, but the integral taken over two of them do vanish. Therefore, the ensemble average of the four terms in equation (4.13) can be obtained as the product of the two ensemble averages:

$$\left\langle \frac{\partial u_z(t)}{\partial z} \int^t u_z(t)\mathrm{d}t \right\rangle \text{ and } \left\langle \frac{\partial u_z(t')}{\partial z} \int^{t'} u_z(t')\mathrm{d}t' \right\rangle. \tag{4.14}$$

Taking the ensemble averages, we shall simply use the standard relation between the ensemble average in x, y, t and in l, m, ω.

We shall describe the motion at the boundary by the sum of the motions of N_{Pl} plumes,

$$u(x, y, t) = \sum_i u_i(x - x_i, y - y_i), \tag{4.15}$$

each of the velocities u_i having the same auto-correlation properties,

$$\langle u_i(x - x_i, y - y_i) u_i(x' - x_i, y' - y_i) \rangle = u^2 \Gamma(x' - x, y' - y). \tag{4.16}$$

We shall consider later the problem of the auto-correlation function. For example, we can assume that

$$\Gamma(x' - x, y' - y) = \Gamma\left(-s^2/b^2\right) \tag{4.17}$$

with

$$s^2 = (x' - x)^2 + (y' - y)^2 \tag{4.18}$$

but we shall assume that in between two plumes there is no correlation. In other words, whereas there is auto-correlation of the motions inside a plume, the motion coming from different plumes can be considered as a white noise.

In order to describe the production of internal waves by random plumes we shall introduce another approximation to the plane parallel description. We have to take into account the fact that we are using in our reasoning the mixture of a plane parallel geometry with spherical properties. To be more coherent, we should use spherical harmonics, but, as we are essentially interested in orders of magnitude, we shall simply assume that the minimum value k_{min} of the wavenumber k is related to the radius r_b of the boundary of the convective zone:

$$k_{min} \cong \left(\frac{2\pi}{\pi r_b}\right) = \left(\frac{2}{r_b}\right). \tag{4.19}$$

In other words, and with the same order of magnitude, we can assume that there is production of plumes inside a square of surface $4L^2$, of the same order of magnitude as the surface of the boundary of the convective zone $4\pi r_b^2$.

We then have

$$\hat{u}(l,m,\omega) = \int\int\int \sum u_i(x - x_i, y - y_i) \mathrm{e}^{ilx+imy+i\omega t} \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}t$$
$$= \int\int\int \sum u_i(x - x_i, y - y_i, t) \mathrm{e}^{il(x-x_i)+im(y-y_i)+i\omega t}$$
$$\times \mathrm{e}^{ilx_i+imy_i}\, \mathrm{d}(x - x_i) \mathrm{d}(y - y_i) \mathrm{d}t. \quad (4.20)$$

Before integration over x and y we have to take the ensemble average

$$\left\langle \left(\sum u_i(x - x_i, y - y_i, t) \mathrm{e}^{ilx_i+imy_i}\right) \left(\sum u_j(x' - x_j, y' - y_j, t) \mathrm{e}^{il'x_j+im'y_j}\right) \right\rangle. \quad (4.21)$$

We now have to take the ensemble average and then carry out the integration in equation (4.13).

In equation (4.11) we call the factor depending on r only $G(r)$:

$$G(r) = \left(\frac{r_b}{r}\right)^{3/2} \left(\frac{\rho_b}{\rho}\right)^{1/2} \left(\frac{N}{N_b}\right)^{-1/2} \quad (4.22)$$

and denote as $F(r)$ the factor $\partial f/\partial r$:

$$F(r) = \frac{\partial f}{\partial r} = -D_{th} \left(\frac{N}{N_b}\right)^3 \left(1 - \frac{\omega^2}{N^2}\right)^{-1/2} \left(\frac{r_b}{r}\right)^3. \quad (4.23)$$

Denoting the parameters of integration for the four functions in (4.13) l_p, m_p (or k_p), ω_p, with $p = 1, 2, 3, 4$ we can write the diffusion coefficient:

$$D_{Mix} = G^4 F^2 \int_0^\infty \mathrm{d}(t' - t) \left\langle \frac{1}{(2\pi)^6} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{k_1^3}{\omega_1^4} \frac{k_3}{\omega_3^4} \frac{1}{\omega_1} \frac{1}{\omega_3} \right.$$
$$\times \exp\left[-\frac{fk_1^3}{\omega_1^4}\right] \exp\left[-\frac{fk_3^3}{\omega_3^4}\right]$$
$$\left. \times \prod_{p=1}^{p=4} \left\{ \frac{\omega_p}{N_b} \hat{u}(l_p, m_p, \omega_p) \mathrm{e}^{i\omega_p t_p} \sum_i \mathrm{e}^{-il_p(x-x_i)-im_p(y-y_i)} \mathrm{d}l_p\, \mathrm{d}m_p\, \mathrm{d}\omega_p \right\} \right\rangle, \quad (4.24)$$

t having the indices 1 and 2 and t' having the indices 3 and 4. Taking into account the assumption expressed by equation (4.14) we calculate the ensemble average, coupling the two products

$$\prod_{p=1,4} \text{ and } \prod_{p=2,3} \quad (4.25)$$

As mentioned above, the turbulent motion due to plumes is characterized by an auto-correlation property which is easily obtained by integration over l, m and ω using the expression for $\hat{u}$ given by equation (4.20):

$$\langle \hat{u}(l,m,\omega)\hat{u}(l',m',\omega')\rangle = N_{Pl}\langle u^2\rangle \left(1/2\pi\right)^3 \delta(l + l')\delta(m + m')\delta(\omega + \omega')$$
$$\int \mathrm{e}^{il\xi}\Gamma(\xi)\mathrm{d}\xi \int \mathrm{e}^{im\eta}\Gamma(\eta)\mathrm{d}\eta \int \mathrm{e}^{i\omega\theta}\Gamma(\theta)\mathrm{d}\theta. \quad (4.26)$$

4.3.1. *Auto-correlation function*

It remains to give the expression for the average correlation function.

Below the level r_{Schw} of the Schwarzschild condition, there is a plume with an upwards motion due to the buoyancy force, starting from the penetration level of a downwards moving plume coming from the convective zone (Larson & Jönssen 1995). The diameter of the buoyant plume goes to infinity and the velocity goes to zero when approaching the critical level.

We can expect a turbulent velocity field distributed around the plume axis, with a mean square velocity and a local auto-correlation width depending on the distance to the plume axis. We have to take the Fourier transform of each local auto-correlation function of the turbulent velocity, then we have to take the average of these local transforms over the whole surface of the buoyant plume.

Finally we assume local auto-correlation functions

$$\Gamma(a, \xi - a)\Gamma(b, \eta - b)\Gamma(\theta) \tag{4.27}$$

where a, b are the horizontal coordinates of the local auto-correlation functions, and ξ and η the distances to these coordinates; θ is a time variable and we shall assume that the average time auto-correlation function is a Gaussian. We need the ensemble average:

$$\langle \Gamma(a, \xi - a)\Gamma(b, \eta - b)\rangle. \tag{4.28}$$

From the properties of a Fourier transform such as

$$\int_{-\infty}^{+\infty} \frac{e^{ikx}}{b^2 + a^2} da = \frac{\pi}{4b^3}(1 + ab)e^{-ab} \tag{4.29}$$

we derive the phenomenological approximation to be used to describe the properties of the average (4.26).

Introducing the notations $\overline{\Gamma}(l)$, $\overline{\Gamma}(m)$ and $\overline{\Gamma}(\omega)$ for the average auto-correlation functions we write the result of the integration to be carried out in equation (4.26):

$$\langle \hat{u}(l, m, \omega)\hat{u}(l', m', \omega')\rangle = N_{Pl}\langle u^2\rangle \delta(l + l')\delta(m + m')\delta(\omega + \omega')b^2\tau(2\pi)^{3/2} \times \overline{\Gamma}(l)\overline{\Gamma}(m)\overline{\Gamma}(\omega). \tag{4.30}$$

The expression for these auto-correlation functions can be derived from the properties of isotropic, uniform turbulence. Lesieur (1993, p. 111) gives the relation between the energy spectrum and the correlation tensor function for three- and two-dimensional turbulence. We can derive as a phenomenological relation the expression for the product : $\overline{\Gamma}(l)\overline{\Gamma}(m)$ will be taken either as $(k/k_M)^{-2}$ or $(k/k_M)^{-3}$. We consider now the asymptotic value of the integration over k, for $(f\tau^4/b_M^3) \ll 1$.

As a first case, we consider the s^{-1} law for the amplitude of the velocity in the spreading flow, as already mentioned. This suggests taking $(k/k_M)^{-2}$ as an approximation of its Fourier transform. We then have

$$\int_{k_{min}}^{\infty} \frac{k^3}{\omega^5}\left(\frac{k_M}{k}\right)^2 \omega\omega' \exp\left\{-\frac{fk^3}{\omega^4}\right\} k\mathrm{d}kr = k_M^2 \frac{\omega}{f}. \tag{4.31}$$

In the second case a faster decrease of the auto-correlation at large distances suggests taking $(k/k_M)^{-3}$ as an approximation of its Fourier transform. We then have

$$\int_{k_{min}}^{\infty} \frac{k^3}{\omega^5}\left(\frac{k_M}{k}\right)^3 \omega\,\omega' \exp\left\{-\frac{fk^3}{\omega^4}\right\} k\mathrm{d}k = C^t k_M^3 \left(\frac{1}{\omega f^2}\right)^{1/3}. \tag{4.32}$$

Taking into account the delta function $\delta(\omega + \omega')$ we can carry out the integration over ω.

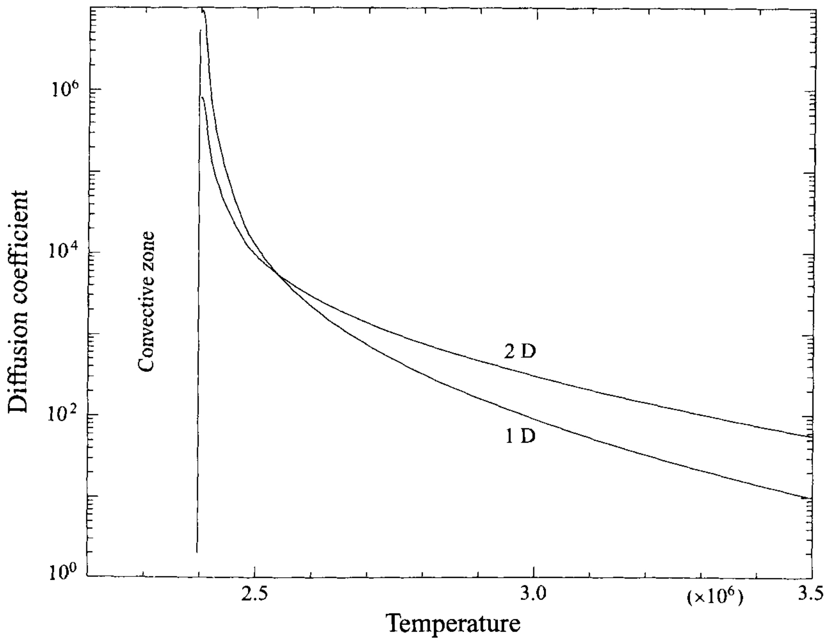

FIGURE 4. Diffusion coefficients given by expressions (4.33) and (4.34).

For the first case, we obtain an expression for the diffusion coefficient similar to the one used by Montalbàn & Schatzman (1995):

$$D_{mix(1)} = \left\{ N_b D_{th} \frac{\rho_b}{\rho} \left(\frac{r_b}{r}\right)^6 u_b^2 N_{Pl} \frac{1}{f\tau} \right\}^2 \tau. \tag{4.33}$$

For the second case we obtain an expression in agreement with former estimates, based on the introduction of a Kolmogorov spectrum (Schatzman & Montalbàn 1995). The diffusion coefficient is

$$D_{mix(2)} = \left\{ 1.79 D_{th} N_b \frac{\rho_b}{\rho} \left(\frac{r_b}{r}\right)^6 u_b^2 N_{Pl} \frac{\tau^{1/3}}{r_b f^{2/3}} \right\}^2 \tau. \tag{4.34}$$

Figure 4 gives the curves for these two diffusion coefficients. They both decrease very quickly down from the convective zone boundary, which suggests the possibility of a good description of lithium deficiency (§5).

4.4. *Large radiative damping*

When approaching the stellar centre, the variable ξ (equation (3.21)) becomes large. We then have the asymptotic expressions

$$a \simeq \left(\frac{\delta}{2}\right)^{1/2} \frac{\omega}{2N} \exp\left(\frac{\xi}{2}\right), \tag{4.35}$$

$$c \simeq -\frac{1}{a}\frac{\delta}{2}\frac{1}{4} \exp\left(\frac{\xi}{4}\right). \tag{4.36}$$

With the expression (3.21) for $\sinh\xi$ we have

$$a \simeq k_b r_b / x, \tag{4.37}$$

$$c \simeq -\left(\frac{\omega r_b^2}{2D_{th}}\right)^{1/2} \frac{N}{\omega}, \tag{4.38}$$

$$\varphi_1 \simeq \frac{3}{2}\frac{1}{x}. \tag{4.39}$$

The velocity is derived from equations (3.1) and (3.2). For a monochromatic wave, we have

$$u, \sim \exp\left[-\mathrm{i}\int\left(\left(\frac{\omega r_b^2}{2D_{th}}\right)^{1/2}\frac{N}{\omega} + k_V r_b\right)\mathrm{d}x\right]\left(\frac{\rho_b}{\rho}\right)^{1/2} x^{k_b r_b} x^{2+3/2} \tag{4.40}$$

where the exponent 2 comes from the relation between ψ and u and the exponent 3/2 comes from equation (4.39). We have then to take the ensemble average defined by equation (4.13). It is proportional to

$$\left\langle \frac{k_b k_b''}{x^2} \exp\left[r_b \ln x\left(k_b + k_b' + k_b'' + k_b'''\right)\right]\right\rangle. \tag{4.41}$$

Neglecting as before the auto-correlation factor $\exp\left(-k_b^2 b^2/4\right)$ the integration over k from k_{min} to ∞ gives

$$D_{Mix} \sim \frac{1}{x^2}\frac{1}{(2r_b \ln x)^2} x^{4k_{min} r_b} x^{14}; \tag{4.42}$$

with $k_{min} = 2/r_b$ we find that

$$D_{Mix} \sim x^{20} (\ln x)^{-2} \quad x \to 0. \tag{4.43}$$

We conclude that the diffusion coefficient vanishes when approaching the stellar centre. This is valid when $x < 0.1$.

5. Diffusion in stellar cores

We still have to consider the validity of the diffusion coefficient close to the centre.

5.1. *Work against gravity*

In a region with a μ gradient, outward diffusion of helium and inward diffusion of hydrogen takes place at the expense of the flux of mechanical energy. The luminosity of mechanical energy of monochromatic waves (Goldreich & Nicholson 1989; Schatzman 1993) is

$$L_E = -4\pi r^2 \rho u_H^2 \frac{\omega}{k_H}\left(\frac{N^2}{\omega^2} - 1\right)^{-1/2}. \tag{5.1}$$

5.2. *First approximation*

We start with the case when the parameter ξ defined in equation (3.21) is assumed to be small. In that case, in equation (5.1) we have to replace the monochromatic velocity by the Fourier transform of the real velocity, including the radiative damping factor, to carry out the integration over the variables k and ω, and finally to take the ensemble average. We then have

$$L_E = 4\pi r^2 \rho \left(\frac{r_b}{r}\right)^2 \left(\frac{\rho_b}{\rho}\right) \left(\frac{1}{N_b}\right)$$
$$\times \left\langle \int_{k_{min}}^{\infty} \mathrm{d}k \int_{-\infty}^{+\infty} \mathrm{d}\omega \exp\left[-\frac{fk^3}{\omega^4}\right] \left(\frac{k_{min}}{k}\right)^2 \omega^2 \right.$$
$$\times \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{+\infty} u(x, y, t) \mathrm{e}^{-\mathrm{i}lx - \mathrm{i}my - \mathrm{i}\omega t} \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}t$$
$$\left. \times \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{+\infty} u(x', y', t') \mathrm{e}^{\mathrm{i}lx' + \mathrm{i}my' + \mathrm{i}\omega t'} \mathrm{d}x'\, \mathrm{d}y'\, \mathrm{d}t' \right\rangle . \tag{5.2}$$

Taking the ensemble average introduces

$$\langle u(x, y, t) u(x', y', t',) \rangle = \langle u^2 \rangle \Gamma(x) \Gamma(y) \Gamma(t) \tag{5.3}$$

the Γ functions being defined by equation (4.16). We have then to carry out the integration

$$\int_{k_{min}}^{\infty} \exp\left[-\frac{fk^3}{\omega^4}\right] \frac{\mathrm{d}k}{k^2} \int_{-\infty}^{+\infty} \omega^2 \exp\left[-\frac{b^2k^2}{4} - \frac{\tau^2\omega^2}{4}\right] \mathrm{d}\omega. \tag{5.4}$$

Assuming as in §4.3 that fk_{min}^3/ω^4 is large and b^2k^2 is small, applying again the method of steepest descent for the integral over ω, and finally fitting the result to the mechanical energy flux at the boundary, we have

$$L_E = 4\pi r^2 \rho \langle u_b^2 \rangle \frac{1}{\tau^2 N_b k_b} N_{Pl} \frac{\pi b^2}{4\pi r_b^2} \exp\left[-\tfrac{3}{4} \left(fk_{min}^3 \tau^4\right)^{1/3}\right]. \tag{5.5}$$

The slow decrease with depth of the luminosity of mechanical energy is essentially due to the interference effect, which decreases the contribution of gravity waves with the largest wavenumber to the radiative damping.

Calling the damping factor ϵ:

$$\epsilon = \exp\left[-\tfrac{3}{4} \left(fk_{min}^3 \tau^4\right)^{1/3}\right], \tag{5.6}$$

we can write the mechanical energy luminosity as

$$L_E = \left(L_E\right)_b \epsilon. \tag{5.7}$$

The mechanical energy which is needed by the diffusion process in the presence of a gradient μ is taken from the energy luminosity and changes the mean square value of the velocity, $\langle u_{\mathrm{V}}^2 \rangle$. In order to express this change, we add a factor $B(r)$ to the expression for L_{E}:

$$L_{\mathrm{E}} = (L_{\mathrm{E}})_b \epsilon B(r). \tag{5.8}$$

Assuming that we have a quasi-stationary situation we write the differential equation governing the flux of mechanical energy as

$$\frac{\mathrm{d}}{\mathrm{d}r} L_{\mathrm{E}} + \frac{1}{\epsilon} \frac{\mathrm{d}\epsilon}{\mathrm{d}r} L_{\mathrm{E}} - 4\pi r^2 \rho \frac{D_{\mathrm{M}}}{H_P} \mu \nabla \mu = 0. \tag{5.9}$$

We call the last term on the left-hand side $G(r)$, and we can write the solution as

$$L_{\mathrm{E}} = (L_{\mathrm{E}})_b\, \epsilon - \epsilon \int_r^{r_1} \frac{1}{\epsilon} G(r) \mathrm{d}r. \tag{5.10}$$

It is then possible to calculate the contribution of the dissipation of mechanical energy to the inward decrease of the amplitude of internal waves. Rewriting (5.10) in a different form, we obtain the expression for $B(r)$:

$$L_{\rm E} = (L_{\rm E})_b\, \epsilon \left\{ 1 - \frac{1}{(L_{\rm E})_l} \int_r^{r_b} \frac{1}{\epsilon} G(r) {\rm d}r \right\} \equiv (L_{\rm E})_b\, \epsilon B(r). \qquad (5.11)$$

We see immediately that a factor B^2 has to be introduced in the diffusion coefficient. From the previous diffusion coefficient $D_{\rm M}$ we can obtain the function $B(r)$:

$$B(r) = 1 - \frac{1}{(L_{\rm E})_b} \int_r^{r_b} \frac{1}{\epsilon} \frac{4\pi r^2 D_{\rm M} B^2 g \mu \nabla \mu}{H_P} {\rm d}r. \qquad (5.12)$$

After differentiation and integration, one obtains the function $B(r)$:

$$\frac{1}{B(r)} = 1 + \frac{1}{(L_{\rm E})_b} \int_r^{r_b} \frac{1}{\epsilon} \frac{4\pi r^2 \rho D_{\rm M} g \mu \nabla \mu}{H_P} {\rm d}r. \qquad (5.13)$$

5.3. *Strong damping*

We deal now with the asymptotic expression when approaching the stellar centre. The expression for the mechanical energy luminosity when ξ is large is

$$L_E \simeq 4\pi^2 \rho \left(\frac{r_b}{r}\right)^2 \left(\frac{\rho_b}{\rho}\right) \frac{1}{N_b} \int {\rm d}k \int \left(\frac{k_{min}}{k}\right)^2 x^{2kr_b} \exp\left[-\left(\frac{\tau^2\omega^2}{4} + \frac{b^2k^2}{4}\right)\right] {\rm d}k. \quad (5.14)$$

Neglecting the wavenumber auto-correlation term, as the main contribution to the integral comes from the small values of k, we obtain the expression

$$L_E \sim \exp\left[2k_{min} r_b \ln x\right]. \qquad (5.15)$$

With the value chosen for k_{min} (equation (4.19)), we have

$$L_E \sim x^4. \qquad (5.16)$$

Applying this result to equation (5.13), it appears that $B(r)$ goes to a finite value when x goes to zero. Therefore, it confirms the property of the diffusion coefficient of going to zero with x.

6. Conclusion

The expression for the diffusion coefficient is obtained by applying the method of ensemble average to random internal waves. It is necessary to introduce from the beginning the Fourier transform of the wave amplitude. The major contribution to the diffusion process is due to low-wavenumber waves, which have the smallest radiative damping.

We have obtained the expression for the horizontal and vertical diffusion coefficients.

The horizontal diffusion coefficient is large, and decreases slowly inwards. We can conclude that this process generates a uniform chemical composition along equipotentials. The vertical diffusion coefficient, due to radiative damping, decreases quickly just below the boundary of the convective zone and appears to have the necessary properties for the understanding of lithium depletion in stars (Montalbàn & Schatzman 1996). At greater depths, a divergence effect, simply due to the geometry of waves propagating inside a spherical body, produces an increase of the diffusion coefficient. Finally, close to the centre, radiative damping becomes the major physical process, and the diffusion coefficient goes to zero. The exact value of the radius where the damping becomes important has not been obtained here, but an order of magnitude value derived from the validity condition of the approximation is given. In the solar case, it is located around $r = 0.1\,R_{\odot}$.

It is clear that the result is dependent on the description of the turbulent motion in the convective zone. But we can nevertheless conclude that macroscopic diffusion due to random internal waves is a physical process which cannot be ignored.

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